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# Simple construction of parameter map germ and its applications(Real Singularities and Real Algebraic Geometry)

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CITATION:

NISHIMURA, TAKASHI. Simple construction of parameter map germ and its applications(Real Singularities and Real Algebraic Geometry). 数理解析研究所講究録 1992, 815: 1-22

ISSUE DATE:

1992-12

URL:

<http://hdl.handle.net/2433/83099>

RIGHT:

# Simple construction of parameter map germ and its applications

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In this note, we shall construct a simple parameter map germ  $(\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^p, 0)$  under the assumption that there is an  $\mathcal{A}$ -morphism (resp. topological  $\mathcal{A}$ -morphism) from a given deformation  $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of a given map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  to the trivial deformation  $f : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$ .

This parameter map germ induces a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from  $\Psi$  to the graph deformation of  $f$ .

By this construction, we can prove the following:

**THEOREM D ([M2]):** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  stable map germs. Suppose there exist a  $C^\infty$  diffeomorphic germ  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)(g \circ s)(x)$ . Then  $f$  and  $g$  are right-left equivalent.

Though our method seems to be close to Martinet's one ([Mr]), we can treat also map germs which are not necessarily  $C^\infty$  stable.

**THEOREM E ([FF]):** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two MT stable map germs. Suppose there exist a  $C^\infty$  diffeomorphic germ  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)(g \circ s)(x)$ . Then  $f$  and  $g$  are topologically right-left equivalent.

**THEOREM A:** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map germs. Suppose there exist a  $C^\infty$  diffeomorphic germ  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map germ

$$M(x) = (m_1(x), \dots, m_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

such that  $f(x) = M(x)(g \circ s)(x)$ . Suppose furthermore there exists a positive integer  $k$  such that

$$m_i(x) - m_i(0) \in m_x^k \mathcal{E}_x^p \subset tf(m_x \mathcal{E}_x^n) + \omega f(m_y \mathcal{E}_y^p)$$

for any  $i$  ( $1 \leq i \leq p$ ). Then  $f$  and  $g$  are right-left equivalent.

As a corollary of theorem A, we get

COROLLARY A: Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ. Suppose there exist positive integers  $k, l$  such that

$$\mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + \omega f(\mathfrak{m}_y \mathcal{E}_y^p)$$

and

$$\mathfrak{m}_x^l \mathcal{E}_x^p \subset tf(\mathfrak{m}_x^2 \mathcal{E}_x^n) + f^* \mathfrak{m}_y \mathfrak{m}_x^k \mathcal{E}_x^p.$$

Then  $f$  is  $(l-1)$ -determined with respect to right-left equivalence.

Corollary A induces the following Gaffney type estimate of the order of determinacy (c.f. [G]).

COROLLARY B: Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ. Suppose there exist positive integers  $k, l$  such that

$$\mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + \omega f(\mathfrak{m}_y \mathcal{E}_y^p)$$

and

$$\mathfrak{m}_x^l \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + f^* \mathfrak{m}_y \mathcal{E}_x^p.$$

Then  $f$  is  $(k+l-1)$ -determined with respect to right-left equivalence.

Corollary B induces the following du Plessis-Wall's estimate of the order of determinacy.

COROLLARY C ([dP, W]): Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ. Suppose there exist a positive integer  $k$  such that

$$\mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + \omega f(\mathfrak{m}_y \mathcal{E}_y^p).$$

Then  $f$  is  $(2k-1)$ -determined with respect to right-left equivalence.

In [W], we can find an estimate of the order of topological determinacy of an MT stable map germ (corollary D bellow) which is due to T. Gaffney, but without proof. By using of our method, we can give a proof of his estimate.

COROLLARY D (GAFFNEY): Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be an MT stable map germ. Suppose there exist a positive integer  $k$  such that

$$\mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + f^*(\mathfrak{m}_y^2) \mathcal{E}_x^p.$$

Then  $f$  is  $k$ -determined with respect to topologically right-left equivalence.

For details on these corollaries, refer to [N].

This note is organized in the following way. In §1 and §2, we give several preparations for the proofs of theorem A, a generalized version of Mather's

classification theorem (theorem D in §5) and the theorem of Fukuda-Fukuda (theorem E in §6). §3 treats algebraic argument which we need for the proof of theorem A. Theorem A will be proved in §4. A generalized version of Mather's classification theorem will be proved in §5. In §6, an alternative proof of the theorem of Fukuda-Fukuda will be given.

The results in this paper are all valid in the complex analytic category as well except example (1.5.2).

#### ACKNOWLEDGEMENT

The original, crude idea of this work was discovered during the author's stay at the University of Liverpool and most of this work was pursued during his stay at the University of North Carolina at Chapel Hill. He would like to thank the Mathematics Departments of both Universities for their hospitality, especially Terry Wall and Jim Damon for their kind encouragement and useful advice. Jim Damon pointed out mistakes of the old version of this work and gave the author a corrected argument in §3. The author would like to give his sincere gratitude to Jim Damon.

#### §1. $\mathcal{K}$ -MORPHISM

##### FROM A GIVEN DEFORMATION TO THE GRAPH DEFORMATION

Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ and  $\Psi_i : (\mathbb{R}^n \times \mathbb{R}^{r(i)}, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  deformation of  $f$  (i.e.  $\Psi_i(x, 0) = f(x)$ ) ( $i = 1, 2$ ).

DEFINITION (1.1). We say if there exist  $C^\infty$  (resp. continuous) map germs  $h : (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0))$ ,  $H : (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0, 0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0, 0, 0))$  and  $\phi : (\mathbb{R}^{r(1)}, 0) \rightarrow (\mathbb{R}^{r(2)}, 0)$  such that the following conditions (1.1.1), (1.1.2), (1.1.3) and (1.1.4) hold, then  $\{h, H, \phi\}$  is a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from  $\Psi_1$  to  $\Psi_2$ .

(1.1.1) the restrictions  $h|_{\mathbb{R}^n \times \{\lambda\}}$  and  $H|_{\mathbb{R}^n \times \{\lambda\} \times \mathbb{R}^p}$  are  $C^\infty$  diffeomorphic (resp. homeomorphic) for any  $\lambda \in \mathbb{R}^{r(1)}$ ,

(1.1.2)  $H(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^{r(2)} \times \{0\}$ ,

(1.1.3) the following diagram commutes:

$$\begin{array}{ccccc}
 (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0, 0, 0)) & \xrightarrow{\pi_{*, \lambda}} & (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(1)}, 0) \\
 H \downarrow & & h \downarrow & & \phi \downarrow \\
 (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0, 0, 0)) & \xrightarrow{\pi_{*, \lambda}} & (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(2)}, 0),
 \end{array}$$

(1.1.4) the following diagram commutes:

$$\begin{array}{ccc}
 (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) & \xrightarrow{(\pi_{\bullet, \lambda}, \Psi_1)} & (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0, 0, 0)) \\
 h \downarrow & & H \downarrow \\
 (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0)) & \xrightarrow{(\pi_{\bullet, \lambda}, \Psi_2)} & (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0, 0, 0)).
 \end{array}$$

Here  $\pi_{\bullet, \lambda}$ ,  $\pi_\lambda$  mean the canonical projection to  $\mathbb{R}^n \times \mathbb{R}^{r(i)}$ ,  $\mathbb{R}^{r(i)}$  respectively. We remark that the conditions (1.1.1), (1.1.2) and (1.1.3) in the definition (1.1) imply  $H(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times (\mathbb{R}^p - \{0\})) \subset \mathbb{R}^n \times \mathbb{R}^{r(2)} \times (\mathbb{R}^p - \{0\})$ ; and the condition (1.1.4) implies  $H(\text{graph}(\Psi_1)) \subset \text{graph}(\Psi_2)$ .

DEFINITION (1.2). We say if there exist  $C^\infty$  (resp. continuous) map germs  $h : (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0))$ ,  $H : (\mathbb{R}^p \times \mathbb{R}^{r(1)}, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^{r(2)}, (0, 0))$  and  $\phi : (\mathbb{R}^{r(1)}, 0) \rightarrow (\mathbb{R}^{r(2)}, 0)$  such that the following conditions (1.2.1) and (1.2.2) hold, then  $\{h, H, \phi\}$  is a  $\mathcal{A}$ -morphism (resp. topological  $\mathcal{A}$ -morphism) from  $\Psi_1$  to  $\Psi_2$ .

(1.2.1) the restrictions  $h|_{\mathbb{R}^n \times \{\lambda\}}$  and  $H|_{\mathbb{R}^p \times \{\lambda\}}$   
are  $C^\infty$  diffeomorphic (resp. homeomorphic)  
for any  $\lambda \in \mathbb{R}^{r(1)}$ ,

(1.2.2) the following diagram commutes:

$$\begin{array}{ccccc}
 (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) & \xrightarrow{(\Psi_1, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^{r(1)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(1)}, 0) \\
 h \downarrow & & H \downarrow & & \phi \downarrow \\
 (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0)) & \xrightarrow{(\Psi_2, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^{r(2)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(2)}, 0).
 \end{array}$$

Let  $\mathcal{G}$  be  $\mathcal{K}$  or  $\mathcal{A}$ . A  $\mathcal{G}$ -morphism (resp. topological  $\mathcal{G}$ -morphism)  $\{h, H, \phi\}$  from  $\Psi_1$  to  $\Psi_2$  is said to be *equivalent* (resp. *topologically equivalent*) if  $\phi$  is  $C^\infty$ -diffeomorphic (resp. homeomorphic). Definitions of  $\mathcal{G}$ -morphism and equivalent  $\mathcal{G}$ -morphism are equivalent to those of Martinet's definitions ([Mr]); and definitions of topological  $\mathcal{G}$ -morphism and topologically equivalent topological  $\mathcal{G}$ -morphism are topological analogues of these. If there exists an equivalent  $\mathcal{A}$ -morphism (resp. topologically equivalent topological  $\mathcal{A}$ -morphism) from a given deformation  $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  to the trivial deformation  $f : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$ , then we say  $\Psi$  has a *triviality* (resp. *topological triviality*).

In this chapter, we show if there is a  $\mathcal{A}$ -morphism (resp. topological  $\mathcal{A}$ -morphism) from a given deformation  $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  to the trivial deformation  $f : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$ , then we can directly construct a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from  $\Psi$  to the graph deformation.

Now suppose there exist  $C^\infty$  (resp. continuous) map germs  $h : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^r, (0, 0))$ ,  $H : (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^r, (0, 0))$  and  $\phi : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^r, 0)$  such that the following (1.3.1) and (1.3.2) hold:

(1.3.1) the restrictions  $h|_{\mathbb{R}^n \times \{\lambda\}}$  and  $H|_{\mathbb{R}^p \times \{\lambda\}}$   
are  $C^\infty$  diffeomorphic (resp. homeomorphic)  
for any  $\lambda \in \mathbb{R}^r$ ,

(1.3.2) the following diagram commutes:

$$\begin{array}{ccccc} (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(\Psi, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\ h \downarrow & & H \downarrow & & \phi \downarrow \\ (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0). \end{array}$$

By (1.3.2), we can write

$$h = (h_1, \phi) \quad \text{and} \quad H = (H_1, \phi).$$

Then, set  $\phi'_H : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^p, 0)$  as

$$\phi'_H(\lambda) = H_1(0, \lambda).$$

Also, set  $h' : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p, (0, 0))$  as

$$h'(x, \lambda) = (h_1(x, \lambda), \phi'_H(\lambda))$$

and set  $H' : (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0, 0, 0))$  as

$$H'(x, \lambda, y) = (h'(x, \lambda), H_1(y, \lambda) - H_1(0, \lambda)).$$

Then we have

(1.4.0)  $h'$  and  $H'$  are  $C^\infty$  (resp. continuous) map germs,

(1.4.1) the restrictions  $h'|_{\mathbb{R}^n \times \{\lambda\}}$  and  $H'|_{\mathbb{R}^n \times \{\lambda\} \times \mathbb{R}^p}$   
are  $C^\infty$  diffeomorphic (resp. homeomorphic)  
for any  $\lambda \in \mathbb{R}^r$ ,

$$(1.4.2) \quad H'(\mathbb{R}^n \times \mathbb{R}^r \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^p \times \{0\},$$

(1.4.3) the following diagram commutes:

$$\begin{array}{ccccc} (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0, 0)) & \xrightarrow{\pi_{\bullet, \lambda}} & (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\ H' \downarrow & & h' \downarrow & & \phi'_H \downarrow \\ (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0, 0, 0)) & \xrightarrow{\pi_{\bullet, y}} & (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) & \xrightarrow{\pi_y} & (\mathbb{R}^p, 0). \end{array}$$

Next, we set  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  as

$$F(x, y) = f(x) - y.$$

We call  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  the graph deformation of  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ .

Then, we can see

$$\begin{aligned} F(h'(x, \lambda)) &= F(h_1(x, \lambda), \phi'_H(\lambda)) && \text{(definition of } h') \\ &= f(h_1(x, \lambda)) - \phi'_H(\lambda) && \text{(definition of } F) \\ &= H_1(\Psi(x, \lambda), \lambda) - \phi'_H(\lambda) && (1.3.2) \\ &= H_1(\Psi(x, \lambda), \lambda) - H_1(0, \lambda) && \text{(definition of } \phi'_H). \end{aligned}$$

Hence, we have

(1.4.4) the following diagram also commutes:

$$\begin{array}{ccc} (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(\pi_{\bullet, \lambda}, \Psi)} & (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0, 0)) \\ h' \downarrow & & H' \downarrow \\ (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) & \xrightarrow{(\pi_{\bullet, y}, F)} & (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0, 0, 0)). \end{array}$$

Therefore,  $\{h', H', \phi'_H\}$  is a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from the given deformation  $\Psi$  to the graph deformation  $F$ .

In particular, by (1.4.2) and (1.4.4) we have

$$(1.4.5) \quad h'(\Psi^{-1}(0)) \subset F^{-1}(0).$$

Furthermore, by (1.4.1) - (1.4.4) and the remark after definition (1.1) we have

$$(1.4.6) \quad h'(\mathbb{R}^n \times \mathbb{R}^r - \Psi^{-1}(0)) \subset \mathbb{R}^n \times \mathbb{R}^p - F^{-1}(0).$$

For the proofs of theorems A, D, E, we need only the properties (1.4.1), (1.4.5) and (1.4.6) (see §. 2).

EXAMPLE (1.5): For any  $C^\infty$  map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ ,

(1) let  $\Psi_1 : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be its  $C^\infty$  deformation of the form  $\Psi_1(x, \lambda) = f(x) + \lambda$ . Then,  $\{h(x, \lambda) = (x, \lambda), H(y, \lambda) = (y - \lambda, \lambda) \text{ and } \phi(\lambda) = \lambda\}$  gives a triviality of  $\Psi_1$ . In this case,  $\phi'_H(\lambda) = -\lambda$ ,  $h'(x, \lambda) = (x, -\lambda)$  and  $H'(x, \lambda, y) = (x, -\lambda, y)$  as we expect. Of course,  $\{h', H', \phi'_H\}$  is an equivalent  $\mathcal{K}$ -morphism from  $\Psi_1$  to  $F$ .

(2) let  $\Psi_2 : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be the deformation of  $f$  of the form  $\Psi_2(x, \lambda) = f(x) - \lambda^3$ ; where  $\lambda^3 = (\lambda_1^3, \dots, \lambda_p^3)$ . Then  $\{h(x, \lambda) = (x, \lambda), H(y, \lambda) = (y + \lambda^3, \lambda) \text{ and } \phi(\lambda) = \lambda\}$  gives a topological triviality of  $\Psi_2$ . In this case,  $\phi'_H(\lambda) = \lambda^3$ ,  $h'(x, \lambda) = (x, \lambda^3)$  and  $H'(x, \lambda, y) = (x, \lambda^3, y)$ . We see  $\{h', H', \phi'_H\}$  is a topologically equivalent topological  $\mathcal{K}$ -morphism from  $\Psi_2$  to  $F$ .

DEFINITION (1.6). Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ and let  $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  deformation of  $f$ . We say  $\Psi$  is  $\mathcal{K}$ -versal (resp. topologically  $\mathcal{K}$ -versal) if for any  $C^\infty$  (resp. continuous) map germs  $h : (\mathbb{R}^n \times \mathbb{R}^t, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^r, (0, 0))$ ,  $H : (\mathbb{R}^n \times \mathbb{R}^t \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0))$  and  $\phi : (\mathbb{R}^t, 0) \rightarrow (\mathbb{R}^r, 0)$  which give a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from  $\tilde{\Psi}$  to  $\Psi$ .

We can define  $\mathcal{A}$ -versality and topological  $\mathcal{A}$ -versality similarly. Let  $\mathcal{G}$  be  $\mathcal{K}$  or  $\mathcal{A}$ . The definition of  $\mathcal{G}$ -versality is equivalent to that of Martinet's definitions ([Mr]); and the definition of topological  $\mathcal{G}$ -versality is its topological analogue.

Since any  $C^\infty$  stable map germ is, when viewed as a  $C^\infty$  deformation of itself,  $\mathcal{A}$ -versal: i.e. any  $C^\infty$  deformation  $\Psi$  of a  $C^\infty$  stable map germ has a triviality; by the above argument we see

THEOREM B (MARTINET([Mr])). For any  $C^\infty$  stable map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ , its graph deformation  $F(x, y) = f(x) - y$  is  $\mathcal{K}$ -versal.

There are several definitions for topological stable map germs (for instance, [dW]). However, it is well-known that for any MT-stable map germ (map germ multi-transversal to Thom-Mather canonical stratification) any  $C^\infty$  deformation of it has a topological triviality (see [M3] or [GWdL]). Hence, again by the above argument, we see

THEOREM C. For any MT-stable map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ , its graph deformation  $F(x, y) = f(x) - y$  is topologically  $\mathcal{K}$ -versal.

## §2. SPECIAL CASE OF §1

In this chapter, we review a part of Martinet's argument in [Mr]. Let  $f, g :$



$(\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be  $C^\infty$  map germs. Suppose there exist a  $C^\infty$  diffeomorphic (resp. homeomorphic) map germ  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map germ

$$M(x) = (m_1(x), \dots, m_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

such that  $f(x) = M(x)(g \circ s)(x)$ .

We set a  $C^\infty$  map germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  as

$$\begin{aligned}\Phi(x, y) &= M(x)((g \circ s)(x) - y) \\ &= f(x) - M(x)y.\end{aligned}$$

Hereafter, we concentrate on studying deformations of this type. Hence, in particular, we assume  $r = p$ . We treat two kinds of  $p$ -dimensional euclidean space  $\mathbb{R}^p$ . When we are considering  $\mathbb{R}^p$  as the target space, we write it  $\mathbb{R}_y^p$ . When we are considering  $\mathbb{R}^p$  as the parameter space, we write it  $\mathbb{R}_\lambda^p$ .

Now suppose there exist  $C^\infty$ -diffeomorphic (resp. homeomorphic) map germs  $h : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0))$ ,  $H : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0))$  and  $\phi : (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_\lambda^p, 0)$  such that the following diagram commutes:

$$\begin{array}{ccccc}(\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}_\lambda^p, 0) \\ h \downarrow & & H \downarrow & & \phi \downarrow \\ (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}_\lambda^p, 0)\end{array}$$

In §1, we defined  $C^\infty$  (resp. continuous) map germs

$$\begin{aligned}\phi'_H &: (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_y^p, 0) \\ h' &: (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_y^p, (0, 0)) \\ H' &: (\mathbb{R}^n \times \mathbb{R}_\lambda^p \times \mathbb{R}_y^p, (0, 0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_y^p \times \mathbb{R}_y^p, (0, 0, 0))\end{aligned}$$

and we saw  $\{h', H', \phi'_H\}$  is a  $\mathcal{K}$ -morphism from  $\Phi$  to  $F$ . By (1.4.5) in §1 and by the form of  $\Phi$ , we have

$$(2.1) \quad f(h_1(x, (g \circ s)(x))) = \phi'_H((g \circ s)(x))$$

as germs at the origin.

We would like to show the following map germ (2.2) is  $C^\infty$  diffeomorphic (resp. homeomorphic) if we assume  $\phi'_H$  is  $C^\infty$  diffeomorphic (resp. homeomorphic).

$$(2.2) \quad x \longmapsto h_1(x, (g \circ s)(x))$$

The map germ (2.2) can be decomposed as follows.

$$(2.3) \quad x \mapsto (x, (g \circ s)(x)) \mapsto h'(x, (g \circ s)(x)) \mapsto h_1(x, (g \circ s)(x)).$$

The first map germ of (2.3) is trivially  $C^\infty$  diffeomorphic. If we assume  $\phi'_H$  is  $C^\infty$  diffeomorphic (resp. homeomorphic), then by (1.4.1) in §1  $h' = (h_1, \phi'_H)$  is  $C^\infty$  diffeomorphic (resp. homeomorphic). Thus, the second map germ of (2.3) is  $C^\infty$  diffeomorphic (resp. homeomorphic). Furthermore, in the case that we assume  $\phi'_H$  is  $C^\infty$  diffeomorphic (resp. homeomorphic), by (1.4.5) and (1.4.6) in §1 we have

$$(2.4) \quad h'(\Phi^{-1}(0)) = F^{-1}(0).$$

By the form of  $\Phi$  and  $F$ , (2.4) means

$$(2.5) \quad \begin{aligned} & \text{the germ of the set } \{h'(x, (g \circ s)(x)) | x \in \mathbb{R}^n\} \\ & = \text{the germ of } F^{-1}(0) \\ & = \text{graph}(f). \end{aligned}$$

By (2.5) and by the form of  $h' = (h_1, \phi'_H)$ , the last map germ of (2.3) is also  $C^\infty$  diffeomorphic.

Therefore, we see

LEMMA (2.6). *If  $\Phi$  has a triviality (resp. topological triviality) and  $\phi'_H$  is  $C^\infty$  diffeomorphic (resp. homeomorphic), then  $f$  and  $g$  are right-left equivalent (resp. topologically right-left equivalent).*

### §3. MODULE

Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ and let

$$M(x) = (m_1(x), \dots, m_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

be also a  $C^\infty$  map germ. Let  $\Phi : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p, 0)$  be the  $C^\infty$  deformation of  $f$  having the following form:

$$\Phi(x, \lambda) = f(x) - M(x)\lambda.$$

In this chapter, we prove the following lemma.

LEMMA (3.1). *Suppose there exists a positive integer  $k$  such that*

$$m_i(x) - m_i(0) \in m_x^k \mathcal{E}_x^p \subset tf(m_x \mathcal{E}_x^n) + \omega f(m_y \mathcal{E}_y^p)$$

for any  $i$  ( $1 \leq i \leq p$ ). Then  $\mathbf{m}_i(\mathbf{x}) - \mathbf{m}_i(0)$  is included in

$$t\tilde{\Phi}_{\mathbf{x}}(\mathbf{m}_{\mathbf{x},\lambda}\mathcal{E}_{\mathbf{x},\lambda}^n) + \omega(\tilde{\Phi}, \pi_{\lambda})(\mathbf{m}_{\mathbf{y},\lambda}\mathcal{E}_{\mathbf{y},\lambda}^p)$$

for any  $i$  ( $1 \leq i \leq p$ ).

PROOF OF LEMMA (3.1): Since we assumed

$$\mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x}}^p \subset tf(\mathbf{m}_{\mathbf{x}}\mathcal{E}_{\mathbf{x}}^n) + \omega f(\mathbf{m}_{\mathbf{y}}\mathcal{E}_{\mathbf{y}}^p),$$

by Malgrange preparation theorem we have

$$(3.2) \quad \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p \subset tf(\mathbf{m}_{\mathbf{x}}\mathcal{E}_{\mathbf{x},\lambda}^n) + \omega(f, \pi_{\lambda})(\mathbf{m}_{\mathbf{y}}\mathcal{E}_{\mathbf{y},\lambda}^p).$$

We set  $\tilde{\Phi} : (\mathbb{R}^n \times \mathbb{R}_{\lambda}^p, (0, 0)) \rightarrow (\mathbb{R}_{\mathbf{y}}^p, 0)$  as

$$\begin{aligned} \tilde{\Phi}(\mathbf{x}, \lambda) &= \Phi(\mathbf{x}, \lambda) + M(0)\lambda \\ &= f(\mathbf{x}) - (M(\mathbf{x}) - M(0))\lambda. \end{aligned}$$

Since we assumed

$$\mathbf{m}_i(\mathbf{x}) - \mathbf{m}_i(0) \in \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x}}^p$$

for any  $i$  ( $1 \leq i \leq p$ ), the difference

$$\tilde{\Phi}(\mathbf{x}, \lambda) - f(\mathbf{x}) = (M(\mathbf{x}) - M(0))\lambda = \sum_{i=1}^p \lambda_i (\mathbf{m}_i(\mathbf{x}) - \mathbf{m}_i(0))$$

is included in

$$\pi_{\lambda}^* \mathbf{m}_{\lambda} \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p \subset (\tilde{\Phi}, \pi_{\lambda})^* \mathbf{m}_{\mathbf{y},\lambda} \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p.$$

Hence, we can approximate (3.2) as follows.

$$\begin{aligned} (3.3) \quad \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p &\subset t\tilde{\Phi}_{\mathbf{x}}(\mathbf{m}_{\mathbf{x}}\mathcal{E}_{\mathbf{x},\lambda}^n) + \omega(\tilde{\Phi}, \pi_{\lambda})(\mathbf{m}_{\mathbf{y}}\mathcal{E}_{\mathbf{y},\lambda}^p) + (\tilde{\Phi}, \pi_{\lambda})^* \mathbf{m}_{\mathbf{y},\lambda} \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p \\ &\subset t\tilde{\Phi}_{\mathbf{x}}(\mathbf{m}_{\mathbf{x},\lambda}\mathcal{E}_{\mathbf{x},\lambda}^n) + \omega(\tilde{\Phi}, \pi_{\lambda})(\mathbf{m}_{\mathbf{y},\lambda}\mathcal{E}_{\mathbf{y},\lambda}^p) + (\tilde{\Phi}, \pi_{\lambda})^* \mathbf{m}_{\mathbf{y},\lambda} \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p. \end{aligned}$$

We set

$$C = \mathcal{E}_{\mathbf{x},\lambda}^p / t\tilde{\Phi}_{\mathbf{x}}(\mathbf{m}_{\mathbf{x},\lambda}\mathcal{E}_{\mathbf{x},\lambda}^n),$$

$$A = \text{image of } \omega(\tilde{\Phi}, \pi_{\lambda})(\mathbf{m}_{\mathbf{y},\lambda}\mathcal{E}_{\mathbf{y},\lambda}^p) \text{ by the canonical projection to } C,$$

$$B = \mathbf{m}_{\mathbf{x}}^k.C.$$

Then, by (3.3) we have

$$(3.4) \quad B \subset A + (\tilde{\Phi}, \pi_{\lambda})^* \mathbf{m}_{\mathbf{y},\lambda} B.$$

Since

$$\begin{aligned} & \dim_{\mathbb{R}} B / (\tilde{\Phi}, \pi_{\lambda})^* \mathfrak{m}_{y,\lambda} B \\ &= \dim_{\mathbb{R}} \mathfrak{m}_x^k \mathcal{E}_x^p / \mathfrak{m}_x^k (t f(\mathfrak{m}_x \mathcal{E}_x^p) + f^* \mathfrak{m}_y \mathcal{E}_x^p) < \infty, \end{aligned}$$

by Malgrange preparation theorem we see  $B$  is finitely generated  $\mathcal{E}_{y,\lambda}$ -module via  $(\tilde{\Phi}, \pi_{\lambda})$ . Hence, by Nakayama's lemma (3.4) implies

$$(3.5) \quad B \subset A$$

From the form  $\tilde{\Phi}(x, \lambda) = \Phi(x, \lambda) + M(0)\lambda$ , we see

$$(3.6) \quad \begin{aligned} & t\tilde{\Phi}_x(\mathfrak{m}_{x,\lambda} \mathcal{E}_{x,\lambda}^n) + \omega(\tilde{\Phi}, \pi_{\lambda})(\mathfrak{m}_{y,\lambda} \mathcal{E}_{y,\lambda}^p) \\ &= t\Phi_x(\mathfrak{m}_{x,\lambda} \mathcal{E}_{x,\lambda}^n) + \omega(\Phi, \pi_{\lambda})(\mathfrak{m}_{y,\lambda} \mathcal{E}_{y,\lambda}^p) \end{aligned}$$

(3.5) and (3.6) yields

$$\mathfrak{m}_i(x) - \mathfrak{m}_i(0) \in \mathfrak{m}_x^k \mathcal{E}_{x,\lambda}^p \subset t\Phi_x(\mathfrak{m}_{x,\lambda} \mathcal{E}_{x,\lambda}^n) + \omega(\Phi, \pi_{\lambda})(\mathfrak{m}_{y,\lambda} \mathcal{E}_{y,\lambda}^p)$$

for any  $i$  ( $1 \leq i \leq p$ ). ■

#### §4. PROOF OF THEOREM A

Let  $\Phi : (\mathbb{R}^n \times \mathbb{R}_{\lambda}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be the  $C^{\infty}$  deformation of  $f$  having the following form:

$$\Phi(x, \lambda) = f(x) - M(x)\lambda.$$

Since

$$\frac{\partial \Phi}{\partial \lambda_i} = -\mathfrak{m}_i(x)$$

for any  $i$  ( $1 \leq i \leq p$ ), by lemma (3.1) we can choose germs of  $C^{\infty}$  vector fields

$$\xi_i \in \mathcal{E}_{x,\lambda}^n \quad \text{and} \quad \eta_i \in \mathcal{E}_{y,\lambda}^p$$

such that

$$(4.1) \quad -\frac{\partial \Phi}{\partial \lambda_i} = \xi_i(\Phi) - \eta_i \circ (\Phi, \pi_{\lambda})$$

$$(4.2) \quad \frac{\partial \Phi}{\partial \lambda_i}(0) = \eta_i(0, 0)$$

for any  $i$  ( $1 \leq i \leq p$ ).

By (4.1), integrating germs of  $C^\infty$  vector fields

$$\xi_1 + \partial/\partial\lambda_1, \dots, \xi_p + \partial/\partial\lambda_p$$

and

$$\eta_1 + \partial/\partial\lambda_1, \dots, \eta_p + \partial/\partial\lambda_p$$

yields  $C^\infty$  diffeomorphic map germs

$$h^{-1} : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0))$$

and

$$H^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0))$$

such that the following diagram commutes.

$$\begin{array}{ccccc} (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}_\lambda^p, 0) \\ h^{-1} \uparrow & & H^{-1} \uparrow & & \parallel \\ (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}_\lambda^p, 0) \end{array}$$

Consider the inverse map germ  $H$  of  $H^{-1}$  and

$$\phi'_H : (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with  $H$ .

Let  $\Theta_i(t; \mathbf{y})$  be the integral curve of  $\eta_i$  starting from  $\mathbf{y}$  and of time  $t$ . Then we can get the image  $\mathbf{y}(\lambda_1, \dots, \lambda_p) = \phi'_H(\lambda_1, \dots, \lambda_p)$  of  $\lambda = (\lambda_1, \dots, \lambda_p)$  by  $\phi'_H$  as the unique solution of the integral equation

$$(4.3) \quad \Theta_1(\lambda_1; \Theta_2(\lambda_2; \dots; \Theta_p(\lambda_p; \mathbf{y}(\lambda_1, \dots, \lambda_p))) \dots) = 0.$$

We differentiate (4.3) with respect to  $\lambda_i$ . Then we get

$$(4.4) \quad \begin{aligned} & \eta_i(\Theta_{i+1}(\lambda_{i+1}; \dots; \Theta_p(\lambda_p; \mathbf{y})) \dots) \\ & + (d\Theta_1)_y \dots (d\Theta_p)_y \partial \mathbf{y}(\lambda_1, \dots, \lambda_p) / \partial \lambda_i = 0 \end{aligned}$$

for any  $i$  ( $1 \leq i \leq p$ ).

Taking values at  $\lambda = 0$  in (4.4), we get

$$\begin{aligned} \frac{\partial \phi'_H}{\partial \lambda_i}(0) &= \frac{\partial \mathbf{y}}{\partial \lambda_i}(0) \\ &= -\eta_i(0, 0) \quad (\mathbf{y}(0, \dots, 0) = 0) \\ &= -\frac{\partial \Phi}{\partial \lambda_i}(0) \quad (4.2) \\ &= \mathbf{m}_i(0). \quad \left( \frac{\partial \Phi}{\partial \lambda_i} = -\mathbf{m}_i \right) \end{aligned}$$

Since  $(\mathbf{m}_1(0), \dots, \mathbf{m}_p(0)) = M(0)$  is in  $GL(p, \mathbb{R})$ ,  $\phi'_H$  is  $C^\infty$  diffeomorphic. Hence, by lemma (2.3),  $f$  and  $g$  are right-left equivalent. ■

### §5. AN ALTERNATIVE PROOF OF MATHER'S CLASSIFICATION THEOREM

In this chapter, we give a proof of the following theorem D which is a generalized version of Mather's classification theorem.

**DEFINITION (5.1).** Let  $X$  be a Banach space. We say a  $C^\infty$  map germ  $f : (X, 0) \rightarrow (\mathbb{R}^p, 0)$  is  $C^\infty$  stable if for any finite dimensional  $C^\infty$  deformation  $\Phi : (X \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$  there exist  $C^\infty$  diffeomorphic map germs

$$\begin{aligned} h &: (X \times \mathbb{R}^r, (0, 0)) \rightarrow (X \times \mathbb{R}^r, (0, 0)) \\ H &: (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \\ \phi &: (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^r, 0) \end{aligned}$$

such that the following diagram commutes:

$$\begin{array}{ccccc} (X \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\ h \downarrow & & H \downarrow & & \phi \downarrow \\ (X \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0). \end{array}$$

**THEOREM D.** Let  $X$  be a Banach space. Let  $f, g : (X, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  stable map germs. Suppose there exists a  $C^\infty$  diffeomorphic germ  $s : (X, 0) \rightarrow (X, 0)$  and a  $C^\infty$  map germ

$$M(x) = (\mathbf{m}_1(x), \dots, \mathbf{m}_p(x)) : (X, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

such that  $f(x) = M(x)(g \circ s)(x)$ . Then  $f$  and  $g$  are isomorphic.

Mather's classification theorem ([M2]) is the case when  $X$  is finite dimensional.

**PROOF:** Let  $M_p(\mathbb{R})$  be the set of all  $(p \times p)$  matrices of real elements and let  $E_p$  be the  $(p \times p)$  unit matrix. For any fixed matrix  $A = (a_1, \dots, a_p) \in M_p(\mathbb{R})$ , define a map germ

$$\Phi_A : (X \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_y^p, 0)$$

as

$$\Phi_A(x, \lambda, B) = f(x) - (A + M(x)B)\lambda.$$

Then  $\Phi_A$  is a finite dimensional  $C^\infty$  deformation of  $f$ . Since  $f$  is  $C^\infty$  stable, for any  $i$  ( $1 \leq i \leq p$ ) and  $A = O$  (zero matrix) we see

$$\sum_{j=1}^p b_{ji} \mathbf{m}_j(\mathbf{x}) = 0 + \sum_{j=1}^p b_{ji} \mathbf{m}_j(\mathbf{x}) = \frac{\partial \Phi_0}{\partial \lambda_i}$$

is included in the set

$$t(\Phi_0)_x(\mathcal{E}_{x,\lambda,B}^n + \omega(\Phi_0, \pi_\lambda, \pi_B)(\mathcal{E}_{y,\lambda,B}^p)).$$

Here we set  $B = [b_{ij}]_{1 \leq i,j \leq p}$ . Since we see trivially

$$t(\Phi_0)_x = t(\Phi_A)_x$$

and

$$\omega(\Phi_0, \pi_\lambda, \pi_B)(\mathcal{E}_{y,\lambda,B}^p) = \omega(\Phi_A, \pi_\lambda, \pi_B)(\mathcal{E}_{y,\lambda,B}^p)$$

for any fixed  $A \in M_p(\mathbb{R})$ , we can choose germs of  $C^\infty$  vector fields

$$\xi_i \in \mathcal{E}_{x,\lambda,B}^n \quad \text{and} \quad \eta_{i,A} \in \mathcal{E}_{y,\lambda,B}^p$$

such that

$$\begin{aligned} -\frac{\partial \Phi_A}{\partial \lambda_i} &= (\mathbf{a}_i + \sum_{j=1}^p b_{ji} \mathbf{m}_j(\mathbf{x})) \\ &= \xi_i(\Phi_A) - \eta_{i,A} \circ (\Phi_A, \pi_\lambda, \pi_B) \end{aligned}$$

for any  $i$  ( $1 \leq i \leq p$ ). Since  $f$  is  $C^\infty$  stable, we can choose germs of  $C^\infty$  vector fields

$$\xi_{jk,A} \in \mathcal{E}_{x,\lambda,B}^n \quad \text{and} \quad \eta_{jk,A} \in \mathcal{E}_{y,\lambda,B}^p$$

such that

$$\begin{aligned} -\frac{\partial \Phi_A}{\partial b_{jk}} &= \lambda_k \mathbf{m}_j(\mathbf{x}) \\ &= \xi_{jk,A}(\Phi_A) - \eta_{jk,A} \circ (\Phi_A, \pi_\lambda, \pi_B) \end{aligned}$$

for any  $j, k$  ( $1 \leq j, k \leq p$ ) and any  $A$  of  $M_p(\mathbb{R})$ .

By integrating

$$\begin{aligned} &\eta_{1,A} + \partial/\partial \lambda_1, \dots, \eta_{p,A} + \partial/\partial \lambda_p, \\ &\eta_{11,A} + \partial/\partial b_{11}, \dots, \eta_{pp,A} + \partial/\partial b_{pp}, \end{aligned}$$

we get a  $C^\infty$  diffeomorphic germ

$$H_A^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)).$$

We consider the map germ

$$\phi'_{H_A} : (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with  $H_A$  and its restriction

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}}$$

for  $B$  sufficiently near the zero matrix.

Let  $\Theta_{i,A}(t; \mathbf{y})$  (resp.  $\Theta_{j,k,A}(t; \mathbf{y})$ ) be the integral curve of  $\eta_{i,A}$  (resp.  $\eta_{j,k,A}$ ) starting from  $\mathbf{y}$  and of time  $t$  for any  $i, j, k$  ( $1 \leq i, j, k \leq p$ ). Then  $\phi'_{H_A}(\lambda_1, \dots, \lambda_p, b_{11}, \dots, b_{pp}) = \mathbf{y}$ , where  $\mathbf{y}$  is the unique solution of the following integral equation

$$\Theta_{1,A}(\lambda_1; \dots; \Theta_{p,A}(\lambda_p; \Theta_{11,A}(b_{11}; \dots (\Theta_{pp,A}(b_{pp}; \mathbf{y}(\lambda_1, \dots, b_{11}, \dots, b_{pp}))) \dots)) = 0.$$

We differentiate this equation with respect to  $\lambda_i$ . Then we have

$$(5.2) \quad \eta_{i,A}(\Theta_{i+1,A}(\lambda_{i+1}; \dots; \Theta_{pp,A}(b_{pp}; \mathbf{y}))) \dots + (d\Theta_{1,A})_{\mathbf{y}} \dots (d\Theta_{pp,A})_{\mathbf{y}} \partial \mathbf{y}(\lambda_1, \dots, \lambda_p, b_{11}, \dots, b_{pp}) / \partial \lambda_i = 0$$

for any  $i$  ( $1 \leq i \leq p$ ).

Taking values at  $\lambda = 0$  and  $B = E_p$  in (5.2), we get

$$(5.3) \quad \begin{aligned} \frac{\partial \phi'_{H_A}}{\partial \lambda_i}(0, E_p) &= \frac{\partial \mathbf{y}}{\partial \lambda_i}(0, E_p) \\ &= -\eta_{i,A}(0, 0, E_p) \\ &= -\frac{\partial \Phi_A}{\partial \lambda_i}(0, 0, E_p) - d(\Phi_A)_x(\xi_i(0, 0, E_p)) \\ &= -\frac{\partial \Phi_A}{\partial \lambda_i}(0, 0, E_p) - df_0(\xi_i(0, 0, E_p)) \\ &= \mathbf{a}_i + \mathbf{m}_i(0) - df_0(\xi_i(0, 0, E_p)) \end{aligned}$$

for any  $i$  ( $1 \leq i \leq p$ ). From (5.3) and since  $\xi_i$  is  $C^\infty$  with respect to  $B = [b_{ij}]$ , we have

LEMMA (5.4). *There exists an open dense subset  $\mathcal{U}$  of  $M_p(\mathbb{R})$  such that for any  $A$  of  $\mathcal{U}$  there exists a neighborhood  $\mathcal{V}_A$  of  $E_p$  in  $M_p(\mathbb{R})$  such that the germ of the restriction*

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}} : (\mathbb{R}_\lambda^p \times \{B\}, (0, B)) \rightarrow (\mathbb{R}_y^p, 0)$$

is  $C^\infty$  diffeomorphic for any  $B$  of  $\mathcal{V}_A$ .

Therefore, by lemmata (2.6) and (5.4), we have



LEMMA (5.5). If we choose  $(p \times p)$  matrix  $A$  of  $\mathcal{U}$  sufficiently near the zero matrix, then  $f(x)$  and  $g_{A,B}(x) = (A + M(x)B)^{-1}f(x)$  are right-left equivalent for any  $B$  of  $\mathcal{V}_A$ .

Next, we take a matrix  $A_0$  of  $\mathcal{U}$  sufficiently near the zero matrix and fix it. We set

$$M(x)^{-1}A_0 = N_{A_0}(x) = (\mathbf{n}_1(x), \dots, \mathbf{n}_p(x)).$$

For any fixed  $B$  of  $\mathcal{V}_{A_0}$ , we define the  $C^\infty$  map germ

$$\tilde{\Phi}_{A_0,B} : (X \times \mathbb{R}_\lambda^p, (0,0)) \rightarrow (\mathbb{R}_y^p, 0)$$

as

$$\tilde{\Phi}_{A_0,B}(x, \lambda) = (N_{A_0}(x) + B)(g_{A_0,B}(x) - \lambda).$$

Then, since

$$\begin{aligned} (g \circ s)(x) &= M(x)^{-1}(A_0 + M(x)B)(A_0 + M(x)B)^{-1}f(x) \\ &= M(x)^{-1}(A_0 + M(x)B)g_{A_0,B}(x) \\ &= (N_{A_0}(x) + B)g_{A_0,B}(x); \end{aligned}$$

we see  $\tilde{\Phi}_{A_0,B}(x, \lambda) = (g \circ s)(x) - (N_{A_0}(x) + B)\lambda$  is a  $C^\infty$  deformation of  $(g \circ s)$ . Since  $(g \circ s)$  is  $C^\infty$  stable, for any  $i$  ( $1 \leq i \leq p$ ) and  $B = E_p$  we see

$$\frac{\partial \tilde{\Phi}_{A_0,E_p}}{\partial \lambda_i} \in t(\tilde{\Phi}_{A_0,E_p})_x(\mathcal{E}_{x,\lambda}^n) + \omega(\tilde{\Phi}_{A_0,E_p}, \pi_\lambda)(\mathcal{E}_{y,\lambda}^p).$$

Since

$$t(\tilde{\Phi}_{A,E_p})_x = t(\tilde{\Phi}_{A,B})_x$$

and

$$\omega(\tilde{\Phi}_{A,E_p}, \pi_\lambda)(\mathcal{E}_{y,\lambda}^p) = \omega(\tilde{\Phi}_{A,B}, \pi_\lambda)(\mathcal{E}_{y,\lambda}^p)$$

for any  $A \in \mathcal{U}$  and  $B \in \mathcal{V}_A$ , we can choose germs of  $C^\infty$  vector fields

$$\tilde{\xi}_i \in \mathcal{E}_{x,\lambda}^n \quad \text{and} \quad \tilde{\eta}_{i,B} \in \mathcal{E}_{y,\lambda}^p$$

such that

$$-\frac{\partial \tilde{\Phi}_{A_0,B}}{\partial \lambda_i} = \tilde{\xi}_i(\tilde{\Phi}_{A_0,B}) - \tilde{\eta}_{i,B} \circ (\tilde{\Phi}_{A_0,B}, \pi_\lambda)$$

for any  $i$  ( $1 \leq i \leq p$ ).

By integrating

$$\tilde{\eta}_{1,B} + \partial/\partial \lambda_1, \dots, \tilde{\eta}_{p,B} + \partial/\partial \lambda_p,$$

we get a  $C^\infty$  diffeomorphic germ

$$H_{A_0, B}^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)).$$

We consider the map germ

$$\phi'_{H_{A_0, B}} : (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with  $H_{A_0, B}$ . We see

$$\begin{aligned} (5.6) \quad \frac{\partial \phi'_{H_{A_0, B}}}{\partial \lambda_i}(0) &= -\tilde{\eta}_{i, B}(0, 0) \\ &= -\frac{\partial \tilde{\Phi}_{A_0, B}}{\partial \lambda_i}(0) - d(\tilde{\Phi}_{A_0, B})_x(\tilde{\xi}_i(0, 0)) \\ &= -\frac{\partial \tilde{\Phi}_{A_0, B}}{\partial \lambda_i}(0) - d(g \circ s)_0(\tilde{\xi}_i(0, 0)) \\ &= \mathbf{n}_i(0) + \mathbf{b}_i - d(g \circ s)_0(\tilde{\xi}_i(0, 0)) \end{aligned}$$

for any  $i$  ( $1 \leq i \leq p$ ). By (5.6), we can choose a matrix  $B$  of  $\mathcal{V}_{A_0}$  with the property that  $\phi'_{H_{A_0, B}}$  is  $C^\infty$  diffeomorphic. Thus, by lemma (2.6), we have

**LEMMA (5.7).** *We can choose a matrix  $B$  of  $\mathcal{V}_{A_0}$  with the property that  $(g \circ s)$  and  $g_{A_0, B}$  are right-left equivalent.*

Lemmata (5.5) and (5.7) concludes that  $f$  and  $g$  are isomorphic. ■

## §6. AN ALTERNATIVE PROOF OF FUKUDA-FUKUDA'S THEOREM

In this chapter, we give a proof of the following theorem.

**THEOREM E ([FF]).** *Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two MT-stable map germs. Suppose there exists a  $C^\infty$  diffeomorphic germ  $s : (\mathbb{R}^n, 0) \rightarrow (X, 0)$  and a  $C^\infty$  map germ*

$$M(x) = (\mathbf{m}_1(x), \dots, \mathbf{m}_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

*such that  $f(x) = M(x)(g \circ s)(x)$ . Then  $f$  and  $g$  are topologically isomorphic.*

For the definition of MT-stable map germs, refer to [M3] or [GWdL]. For our proof of theorem E, we use only the following fact on MT-stable map germs.

FACT (6.1). Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be an MT-stable map germ. Then for any  $C^\infty$  deformation  $\Phi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$  there exist Whitney stratifications  $S$  of  $\mathbb{R}^n \times \mathbb{R}^r$  and  $T$  of  $\mathbb{R}^p \times \mathbb{R}^r$  such that the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \xrightarrow{(\Phi, \pi_\lambda)} (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \xrightarrow{\pi_\lambda} (\mathbb{R}^r, 0)$$

is a Thom sequence with respect to  $S, T$  and  $\{\mathbb{R}^r\}$ .

PROOF OF THEOREM E: As in §5, for any fixed matrix  $A = (a_1, \dots, a_p) \in M_p(\mathbb{R})$ , define a map germ

$$\Psi_A : (X \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_y^p, 0)$$

as

$$\Psi_A(x, \lambda, B) = f(x) - (A + M(x)B)\lambda.$$

Then  $\Psi_A$  is a  $C^\infty$  deformation of  $f$ . Since  $f$  is MT-stable, by (6.1), there exist Whitney stratifications  $S$  of  $\mathbb{R}^n \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R})$  and  $T$  of  $\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R})$  such that the germ of the sequence

$$\begin{array}{c} (\mathbb{R}^n \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \downarrow (\Psi_0, \pi_{\lambda, B}) \\ (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \downarrow \pi_{\lambda, B} \\ (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \end{array}$$

is a Thom sequence with respect to  $S, T$  and  $\{\mathbb{R}_\lambda^p \times M_p(\mathbb{R})\}$ .

For any stratum  $T$  of  $T$  and any  $A$  of  $M_p(\mathbb{R})$ , we set

$$T_A = \{(y, \lambda, B) - (A\lambda, \lambda, B) \in \mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}) \mid (y, \lambda, B) \in T\}$$

and

$$T_A = \{T_A\}.$$

Then, since

$$\begin{aligned} \Psi_A(x, \lambda, B) &= \Psi_0(x, \lambda, B) - A\lambda \\ &= \Psi_0(x, \lambda, B) + (\text{family of parallel translation of } \mathbb{R}_y^p) \end{aligned}$$

we see

LEMMA (6.2). For any matrix  $A$  of  $M_p(\mathbb{R})$ , the germ of the sequence

$$\begin{array}{c} (\mathbb{R}^n \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \downarrow (\Psi_A, \pi_{\lambda, B}) \\ (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \downarrow \pi_{\lambda, B} \\ (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \end{array}$$

is a Thom sequence with respect to  $S, T_A$  and  $\{\mathbb{R}_\lambda^p \times M_p(\mathbb{R})\}$ .

By (6.2) we see

LEMMA (6.3). There exists an open dense subset  $\mathcal{U}$  of  $M_p(\mathbb{R})$  such that for any  $A$  of  $\mathcal{U}$  there exists a neighborhood  $\mathcal{V}_A$  of  $E_p$  in  $M_p(\mathbb{R})$  such that the subset  $\{0\} \times \mathbb{R}_\lambda^p \times \{B\} (\subset \mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}))$  is transversal to the intersection  $T_A \cap (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times \{B\})$  near  $\{0\} \times \{0\} \times \{0\}$  for any  $B$  of  $\mathcal{V}_A$  and any  $T_A$  of  $\mathcal{T}_A$ .

By lifting vector fields  $\partial/\partial\lambda_1, \dots, \partial/\partial\lambda_p, \partial/\partial b_{11}, \dots, \partial/\partial b_{pp}$ , we get germs of vector fields

$$\begin{array}{l} \eta_{1,A} + \partial/\partial\lambda_1, \dots, \eta_{p,A} + \partial/\partial\lambda_p, \\ \eta_{11,A} + \partial/\partial b_{11}, \dots, \eta_{pp,A} + \partial/\partial b_{pp}, \end{array}$$

which are stratified with respect to the stratification  $\mathcal{T}_A$  and satisfy the control conditions. By integrating these stratified vector fields, we get a homeomorphic germ

$$H_A^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)).$$

We consider the map germ

$$\phi'_{H_A} : (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with  $H_A$  and its restriction

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}}$$

for  $B$  sufficiently near the zero matrix.

Let  $\Theta_{i,A}(t; y)$  (resp.  $\Theta_{j,k,A}(t; y)$ ) be the integral curve of  $\eta_{i,A}$  (resp.  $\eta_{j,k,A}$ ) starting from  $y$  and of time  $t$  for any  $i, j, k$  ( $1 \leq i, j, k \leq p$ ). Then

$\phi'_{H_A}(\lambda_1, \dots, \lambda_p, b_{11}, \dots, b_{pp})$  can be given as the unique solution of the following integral equation

$$\Theta_{1,A}(\lambda_1; \dots; \Theta_{p,A}(\lambda_p; \Theta_{11,A}(b_{11}; \dots (\Theta_{pp,A}(b_{pp}; \phi'_{H_A}(\lambda_1, \dots, b_{pp})) \dots)) = 0.$$

Since the germs of vector fields

$$\begin{aligned} \eta_{1,A} + \partial/\partial\lambda_1, \dots, \eta_{p,A} + \partial/\partial\lambda_p, \\ \eta_{11,A} + \partial/\partial b_{11}, \dots, \eta_{pp,A} + \partial/\partial b_{pp}, \end{aligned}$$

are controlled, by lemma (6.3), we see

LEMMA (6.4). For any  $A$  of  $\mathcal{U}$  and any  $B$  of  $\mathcal{V}_A$ , the germ of the restriction

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}} : (\mathbb{R}_\lambda^p \times \{B\}, (0, B)) \rightarrow (\mathbb{R}_y^p, 0)$$

is injective.

Since  $\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}}$  is continuous, injectivity means being homeomorphic. Therefore, by lemma (2.6) we have

LEMMA (6.5). If we choose  $(p \times p)$  matrix  $A$  of  $\mathcal{U}$  sufficiently near the zero matrix, then  $f(x)$  and  $g_{A,B}(x) = (A + M(x)B)^{-1}f(x)$  are topologically right-left equivalent for any  $B$  of  $\mathcal{V}_A$ .

Next, we take a matrix  $A_0$  of  $\mathcal{U}$  sufficiently near the zero matrix and fix it. We set

$$M(x)^{-1}A_0 = N_{A_0}(x) = (n_1(x), \dots, n_p(x)).$$

For any fixed  $B$  of  $\mathcal{V}_{A_0}$ , we define the  $C^\infty$  map germ

$$\tilde{\Psi}_{A_0,B} : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p, 0)$$

as

$$\tilde{\Psi}_{A_0,B}(x, \lambda) = (N_{A_0}(x) + B)(g_{A_0,B}(x) - \lambda).$$

Then, since

$$\begin{aligned} (g \circ s)(x) &= M(x)^{-1}(A_0 + M(x)B)(A_0 + M(x)B)^{-1}f(x) \\ &= M(x)^{-1}(A_0 + M(x)B)g_{A_0,B}(x) \\ &= (N_{A_0}(x) + B)g_{A_0,B}(x); \end{aligned}$$

we see  $\tilde{\Psi}_{A_0,B}(x, \lambda) = (g \circ s)(x) - (N_{A_0}(x) + B)\lambda$  is a  $C^\infty$  deformation of  $(g \circ s)$ . Since  $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  is MT-stable and  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  is  $C^\infty$

diffeomorphic,  $(g \circ s)(x) = M(x)^{-1}f(x)$  is MT-stable. Thus, by (6.1), there exist Whitney stratifications  $\tilde{S}$  of  $\mathbb{R}^n \times \mathbb{R}_\lambda^p$ ,  $\tilde{T}$  of  $\mathbb{R}_y^p \times \mathbb{R}_\lambda^p$  such that the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \xrightarrow{(\tilde{\Psi}_{A_0, 0}, \pi_\lambda)} (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \xrightarrow{\pi_\lambda} (\mathbb{R}_\lambda^p, 0)$$

is a Thom sequence with respect to  $\tilde{S}, \tilde{T}$  and  $\{\mathbb{R}_\lambda^p\}$ .

For any stratum  $\tilde{T}$  of  $\tilde{T}$  and any  $B$  of  $M_p(\mathbb{R})$ , we set

$$\tilde{T}_B = \{(y, \lambda) - (B\lambda, \lambda) \in \mathbb{R}_y^p \times \mathbb{R}_\lambda^p \mid (y, \lambda) \in \tilde{T}\}$$

and

$$\tilde{T}_B = \{\tilde{T}_B\}.$$

Then, since

$$\tilde{\Psi}_{A_0, B}(x, \lambda) = \tilde{\Psi}_{A_0, 0}(x, \lambda) - B\lambda$$

we see

LEMMA (6.6). For any matrix  $B$  of  $M_p(\mathbb{R})$ , the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \xrightarrow{(\tilde{\Psi}_{A_0, B}, \pi_\lambda)} (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \xrightarrow{\pi_\lambda} (\mathbb{R}_\lambda^p, 0)$$

is a Thom sequence with respect to  $\tilde{S}, \tilde{T}_B$  and  $\{\mathbb{R}_\lambda^p\}$ .

By lemma (6.6) and by using the same argument as before, we can choose a matrix  $B$  of  $\mathcal{V}_{A_0}$  with the property that  $\phi'_{H_{A_0, B}}$  is homeomorphic. Thus, by lemma (2.6), we have

LEMMA (6.7). We can choose a matrix  $B$  of  $\mathcal{V}_{A_0}$  with the property that  $(g \circ s)$  and  $g_{A_0, B}$  are topologically right-left equivalent.

Lemmata (6.6) and (6.7) concludes that  $f$  and  $g$  are topologically isomorphic. ■

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